In the motion of a satellite near the earth, an important force, in addition to the gravitational force, is that due to atmospheric drag. If it is assumed that the drag force is tangential to the path and proportional to the density and to the square of the speed, then the basic equations in the original variables can be written as

$$\begin{aligned} d^2r/dt^2 - r(d\vartheta/dt)^2 + K/r^2 &= (-C_DA/2m)\rho(dr/dt)[(dr/dt)^2 + r^2(d\vartheta/dt)^2]^{1/2} \\ (1/r)(d/dt)(r^2d\vartheta/dt) &= (-C_DA/2m)\rho r(d\vartheta/dt)[(dr/dt)^2 + r^2(d\vartheta/dt)^2]^{1/2} \end{aligned} \quad (2)$$

where $\rho$ is the density, $m$ the mass of the satellite, $A$ the normal cross-sectional area, and $C_D$ the drag coefficient assumed as constant.

If the satellite is sufficiently high, then the drag terms may be dropped, and the analysis leads to the usual Kepler results; if the satellite is near the re-entry condition, then the drag forces will dominate the gravitational forces, and simplifying assumptions can be made.[4] However, in the studies of the lifetime of a satellite, or in orbital studies where it is assumed that the satellite is several revolutions away from the re-entry condition, the accurate inclusion of the drag terms becomes necessary.

Roberson[5] analyzed Eqs. (2) by a formal perturbation procedure after the introduction of new variables. Since the quantity $z$ in Eq. (1) is constant in the drag-free case, it is to be expected that this quantity would vary slowly as a function of the time in the presence of drag forces. Roberson used $R/r$ and $KR/(r^2d\vartheta/dt)^2$ as new variables and was able to reduce the original equations to a second-order equation that was linear and a first-order equation. However, Roberson used only one dynamical invariant in the analysis. It has been noted that there are two components of velocity which are invariant in the two-body motion. Without any essential restriction in generality, the dynamical equations can be written for the case of constant tangential thrust; the modifications in the case of drag are clear.

It is of interest to change to dimensionless variables; the dimensionless time $\tau = (R/g)^{1/2}t$, where $g$ is the acceleration due to gravity at the surface of a spherical earth, and the dimensionless velocity is $V$, where $v = (gR)^{1/2}V$. The nondimensional measure of the thrust may be denoted by $\mu$. The dynamical equations in (2) may be written as

$$\begin{aligned} d^2p/d\tau^2 - p(d\vartheta/d\tau)^2 + (1/p^2) &= \mu(dp/d\tau)/V \\ p(d^2\vartheta/d\tau^2) + 2(dp/d\tau)(d\vartheta/d\tau) &= \mu p(d\vartheta/d\tau)/V \end{aligned} \quad (3)$$

If the definitions used in Eqs. (1) are used in Eqs. (3), one may write

$$\begin{aligned} p &= z/(x + y\cos\varphi) \\ dp/d\tau &= y\sin\vartheta \\ d\vartheta/d\tau &= (x + y\cos\vartheta)^2/z \end{aligned} \quad (4)$$

The first relation in Eqs. (4) shows that, for constant $(x,y,z)$, $p$ is in the correct polar form for the equation of the undisturbed orbit. In the Kepler case these constants are known, and the orbit is fixed. In the case of small thrust or drag, $(x,y,z)$ should be slowly varying functions of $\vartheta$ and therefore of the time $t$. There are several advantages to be gained from working with the orbital equations in the form of Eqs. (4); the first equation, for example, gives the instantaneous ellipse at any instant of time if $(x,y,z)$ are known at that time. Also $(y/x)$ is the instantaneous eccentricity of the orbit. The functions $(x,y,z)$ can be determined successively from the solution of first-order equations; in each case the starting approximations are known constants.

To indicate the work briefly, the first and second equations of (4) must be related; if primes denote differentiations with respect to the angle $\vartheta$, this relation can be written as

$$x'z + y'z\cos(\vartheta) - z'(x + y\cos\vartheta) = 0 \quad (5)$$

The second relation is determined conveniently by dividing the left and right sides, respectively, of Eqs. (3). This equation ultimately[3] can be written as

$$y'(-z\sin\vartheta) + z'(-y\sin\vartheta) = 1 - xz \quad (6)$$

Equation (5), of course, does not depend upon dynamical considerations; in the Kepler case $xz = 1$, and the right side of Eq. (6) vanishes for the starting approximation. This equation does not depend upon the force law, but the force is required to remain tangential to the orbit. This equation does not involve the density, magnitude of the velocity, or any thrust parameter. The third and final equation does involve the thrust parameter:

$$z' = (\mu z^2/V)/(x + y\cos\vartheta)^2 \quad (7)$$

The three first-order equations in $(x,y,z)$ as functions of $\vartheta$ are Eqs. (5–7).

It now is assumed that the solutions may be written in the form of the perturbation series solution:

$$\begin{aligned} x(\vartheta) \quad & x_0 + \mu x_1(\vartheta) + \mu^2x_2(\vartheta) + \ldots \\ y(\vartheta) \quad & y_0 + \mu y_1(\vartheta) + \mu^2y_2(\vartheta) + \ldots \\ z(\vartheta) \quad & z_0 + \mu z_1(\vartheta) + \mu^2z_2(\vartheta) + \ldots \end{aligned} \quad (8)$$

and it is known that, if $\mu = 0$, $x,y$, and $z$ are constants [Eqs. (1)]. The terms with subscripts unity then can be written down in terms of integrals of known functions.

### References

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# Comments

## Errata

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THE authors would like to call attention to the following misprints that appeared in the paper "On Calculations of the Laminar Separation Point, and Results for Certain Flows," by Morris Morduchow and Stanley P. Reyle, in the Readers' Forum of the *Journal of the Aerospace Sciences*, August 1962, p. 996.

In Eq. (2), the exponent should read "1/(6.13$n$-1)." In Eq. (3), the exponent should read "1/(6.13$n$ + 1)." In the fourth line after Eq. (3), the beginning of the sentence should read "For $u_1/u_\infty = 1 - \xi^n (n > 0)$ . . . . "

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